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GENERAL THEORY OF THE SOLUTIONS OF THE EQUATIONS OF MOTION OF AN ELASTIC MEDIUM OF DIFFERENT MODULI*

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The equation to be investigated

$$u_{tt} = (u_x - a\,|\,u_x\,|)_x, \quad a = \text{const} \tag{0.1}$$

for $|a| < 1$ describes one-dimensional longitudinal motions of an elastic medium of different moduli /1, 2/. For $a = 1$ describes analogous motions of an elastic granular medium, i.e., a medium having a finite (positive) modulus under compression and exerting no resistance to tensile forces. To be specific, the case $0 \leqslant a \leqslant 1$ is examined.

For the case when $0 < a < 1$ the kinds of discontinuities in the solution are classified (shocks, signotons, semisignotons, simple discontinuities), and the concept of a local solution is introduced, as describing the simplest qualitative structures of discontinuous solutions (189 such structures). By piecing together the local solutions we can find the global solution. The process by which discontinuities occur in the solutions and their bifurcation are investigated.

A general theory of solutions is constructed for Eq.(0.1) in an analogous manner for $a = 1$. In addition to the listed kinds of discontinuities, a new kind occurs here, a discontinuity in the continuity of displacement (spall). Specific problems of wave reflection from a free edge and from a rigid wall are considered in which distinctive, substantially non-linear, effects appear.

Equation (0.1) is a special case of the equation

$$u_{tt} - (\varphi(u_x))_x = 0 \tag{0.2}$$

to whose investigation many papers are devoted /3/. A number of facts are known about (0.2) which distinguish it from linear second-order hyperbolic equations. For instance, the Cauchy problem for (0.2) with infinitely differentiable initial functions and $\varphi(\lambda), \varphi'(\lambda) > 0$ cannot have solutions with continuous first- and second-order derivatives in the large, i.e., for all $t > 0$ /4/. The generalized solution of the Cauchy problem for (0.2) exists in the large /5/, but it is not generally unique. These facts also hold for (0.1). A deeper analysis of the solutions can be performed in the case of (0.1) as compared with (0.2) of general form, and in particular, various versions of the discontinuities that orginate in the solution of (0.1) and their bifurcations can be investigated in detail.

The investigation of (0.1) is of interest both in connection with the general theory of non-linear hyperbolic equations and in connection with the fact that problems in the theory of elastic bodies of different moduli, elastic-plastic media /6/, phase transitions /7/, and geophysics problems /8/ result in (0.1).

It is of interest to investigate the equation

$$u_{tt} = (u_x - |\,u_x\,|)_x + A, \quad A = \text{const} \tag{0.3}$$

which describes the motion of a particle of an elastic-granular medium in a gravity force field. It turns out that the solutions of (0.3), in addition to the discontinuities inherent in the solutions of (0.1), can have yet another kind of discontinuity, a break in the continuity (spall). Several specific problems with a clear physical interpretation exhibiting a substantial difference in the solutions as compared with the analogous linear formulation of the problem are solved for (0.3). These soltuions also show a manifold of qualitative effects described by (0.3).

Note that, by analogy with (0.1), a general theory of solutions can be constructed for the equation

$$u_{tt} = (u_x - a\,|\,u_x\,|)_x + A, \quad |\,a\,| < 1, \quad A = \text{const} \tag{0.4}$$

Solutions of (0.3) are limits of the solutions of (0.4) as $a \to 1, a < 1$. However, (0.4) is more complex than (0.3). Consequently, as will be done in Sect.6, it is natural to construct a general theory of solutions directly for (0.3). Moreover, the solutions of (0.3) can be considered as asymptotic forms of the solutions of (0.4) for $a \sim 1$.

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Equations of the form of (0.1) and (0.3) are of methodological importance since they enable a fairly detailed mathematical investigation to be made of a large number of non-self-similar problems on the one hand, and have a natural physical interpretation on the other. The mathematical invesigation of these equations is quite fruitful and is far from being exhausted in the present paper.

1. Generalized solution of equation (0.2). Piecewise-smooth solutions. Hugoniot conditions. No-growth conditions for the mechanical energy. A function $u(x, t)$, such that the functions $u_t, \varphi(u_x)$ are locally summable and

$$\int_{\Omega} (u_t h_t - \varphi(u_x) h_x)\, dx\, dt = 0, \quad \forall h(x, t) \in C_0^{\infty}(\Omega) \tag{1.1}$$

is called a generalized solution of (0.2) in the domain Ω.

The definition (1.1) for the generalized solution corresponds to the Hamilton principle /9/. Later Ω has the form $p < x < q,\ 0 \leqslant t \leqslant T < \infty$, where p, q can be $-\infty, \infty$ respectively.

Let us consider a finite number of smooth curves in

$$\Gamma_i = \{x, t\colon x = x_i(t), 0 \leqslant t \leqslant T\}, i = 1, \ldots, N$$

that intersect each other at not more than a finite number of points. The function $u(x, t)$ is called piecewise-smooth in Ω if it is continuous in Ω, has uniformly bounded derivatives u_x, u_t outside the lines Γ_i that are uniformly continuous in any simply-connected open sub-domain Ω' that does not intersect $\Gamma_i, i = 1, \ldots, N$ and whose boundary can have a non-empty intersection with just one of the Γ_i. In addition, it is assumed that locally summable functions $u_{tt}, \varphi(u_x)_x$ exist in the domain $\Omega \setminus \Gamma, \Gamma = U\Gamma_i$.

From (1.1) integration by parts yields.

Theorem 1. A piecewise-smooth function is a generalized solution of (0.2) if and only if (0.2) is satisfied almost everyhwere in Ω and the equations

$$[(x_i'(t))\, u_x - \varphi(u_x)]_{x_i(t)} = 0, \ [u]_{x_i(t)} = 0 \tag{1.2}$$
$$([f]_{x(t)} = f(x(t) + 0, t) - f(x(t) - 0, t) = f^+ - f^-)$$

are valid on Γ_i.

Equations (1.2) are called the Hugoniot conditions. It is known /3/ that the solution of the Cauchy problem for (0.2) is not generally unique in the class of piecewise-smooth functions. In this connection, we impose an additional constraint on the solution, corresponding to the requirement of local non-growth of the mechanical energy in the medium, and having the form

$$\int_{\gamma} \left(\frac{u_t^2}{2} + \Phi(u_x)\right) dx + \varphi(u_x) u_t\, dt \geqslant 0, \quad \Phi(0) = 0, \quad \Phi' = \varphi \tag{1.3}$$

where γ is a contour in Ω transversal to Γ_i oriented counter-clockwise to the motion (the t axis is directed upward, and the x axis to the right).

We will clarify the inequality (1.3). We consider the equation of motion of a visco-elastic medium corresponding to (0.2)

$$u_{tt}^{\mu} - \varphi(u_x^{\mu})_x - \mu\psi(u_x^{\mu})_{xt} = 0, \ \psi'(\lambda) \geqslant 0$$

The solutions $u^{\mu}(x, t)$ are assumed to be fairly smooth functions. The equation

$$\int_{\gamma} \left(\frac{(u_t^{\mu})^2}{2} + \Phi(u_x^{\mu})\right) dx + (\varphi(u_x^{\mu}) + \mu\psi(u_x^{\mu})_t) u_t^{\mu}\, dt = \iint_{D} \mu\psi'(u_x^{\mu})(u_{xt}^{\mu})^2\, dx\, dt \tag{1.4}$$

(D is the domain bounded by the contour γ) follows from Green's formula for the solution $u^{\mu}(x, t)$.

As $\mu \to 0$, the solution $u^{\mu}(x, t)$ tends to the solution $u(x, t)$ of Eq.(0.2) for which there may be discontinuities of the first kind in the first derivatives, i.e., u_x^{μ} and u_t^{μ} are bounded while u_{xt}^{μ} has the form of a δ-like family of functions on the line of discontinuities of the solution of (0.2). Consequently, since the contour γ is transversal to Γ_i, we have

$$\int_{\gamma} \mu\psi(u_x^{\mu})_t u_t^{\mu}\, dt \to 0 \quad \text{as} \quad \mu \to 0$$

Passing to the limit as $\mu \to 0$ in (1.4), we obtain the inequality (1.3).

Theorem 2. Let $u(x, t)$ be a piecewise-smooth function satisfying the Hugoniot conditions on the line $x = x(t), t_1 < t < t_2$, where $x'(t) \neq 0,\ [u_x]_{x(t)} \neq 0$. It then follows from (1.3) that for $t_1 < t < t_2$

$$\operatorname{sign}[u_x]_{x(t)} \operatorname{sign} x'(t) = \operatorname{sign}\left(\varphi\left(\frac{\lambda_1+\lambda_2}{2}\right) - \frac{1}{2}(\varphi(\lambda_1)+\varphi(\lambda_2))\right) \tag{1.5}$$

where $\lambda_1\ (\lambda_2)$ lies in the neighbourhood of the point $u_x^+\ (u_x^-)$.

To be specific we henceforth limit ourselves to the case when $0 < a < 1$.

It follows from Theorem 2 that if $u(x, t)$ is a piecewise-smooth solution of (0.1), and $u_x^+u_x^- < 0$, then in the case of a local maximum (minimum) in x in $x(t)$ we have $x'(t) < 0$ ($x'(t) > 0$). Conditions (1.5) for Eq.(0.2) agree with the conditions for stability of discontinuities /10/.

2. Classification of discontinuities of a piecewise-smooth solution of Equation 0.1. Propagation velocities of the discontinuities. Integrability of the Hugoniot conditions. Let $u(x, t)$ be a piecewise-smooth solution of (0.1). We call the smooth line $x = x(t)$ a line of discontinuity if u_x has a jump on this line or u_x changes sign when crossing it. There are four kinds of discontinuities of the solution (we denote them by $\alpha, \beta, \gamma, \delta$), which are defined as follows: α (shock): $u_x^+u_x^- < 0$; β (signoton): $u_x^+ = u_x^- = 0$, u_x changes sign on crossing the line $x = x(t)$; γ (semi-signoton) $u_x^+u_x^- = 0$, $u_x^+ \neq u_x^-$, u_x changes sign on crossing the line $x = x(t)$; δ (simple discontinuity): $u_x^+ \neq u_x^-$, u_x retains its sign as it crosses $x = x(t)$.

Let u be a piecewise-smooth solution of (0.1), and let $x = x(t)$ be a line of discontinuity for u. Then u allows of the representation

$$\begin{aligned} u(x, t) &= p_1(x + bt) + q_1(x - bt),\ u_x \leqslant 0 \\ u(x, t) &= p_2(x + ct) + q_2(x - ct),\ u_x \geqslant 0 \end{aligned} \tag{2.1}$$

Here and below we have used the notation $b = \sqrt{1 + a}$, $c = \sqrt{1 - a}$.

In this case the Hugoniot conditions have the form

$$\begin{gathered} (p_1'(x(t) + bt) + q_1'(x(t) - bt))((x'(t))^2 - b^2) = \\ (p_2'(x(t) + ct) + q_2'(x(t) - ct))((x'(t))^2 - c^2) \\ p_1(x(t) + bt) + q_1(x(t) - bt) = p_2(x(t) + ct) + q_2(x(t) - ct) \end{gathered} \tag{2.2}$$

When $x = x(t)$ is a shock front, we have from the first equation in (2.2)

$$\frac{p_1'(x(t) + bt) + q_1'(x(t) - bt)}{p_2'(x(t) + ct) + q_2'(x(t) - ct)} = \frac{(x'(t))^2 - c^2}{(x'(t))^2 - b^2} < 0 \tag{2.3}$$

Inequality (2.3) shows that the inequalities

$$b > |x'(t)| > c \tag{2.4}$$

are satisfied for the velocity of its motion in the case of a shock.

Let $x = x(t)$ be a signoton front. It follows from (2.1) that

$$bp_1'(x(t) + bt) = -bq_1'(x(t) - bt) = -cq_2'(x(t) - ct) = cp_2'(x(t) + ct) \tag{2.5}$$

It can be shown that if $u_{xx}^{\pm}$ exist, then $u_{xx}^+u_{xx}^- > 0$.

We express $u_{xx}^{\pm}$ from Eqs.(2.5)

$$\begin{gathered} (p_1''(x(t) + bt) - q_1''(x(t) - bt))(x'(t) - b) = \\ 2bq_1''(x(t) - bt); \\ 2b(x'(t) - b)q_1''(x(t) - bt) = (p_2''(x(t) + ct) - \\ q_2''(x(t) - ct))(x'(t))^2 - c^2) \end{gathered} \tag{2.6}$$

It follows from (2.6) that

$$\frac{p_1''(x(t) + bt) - q_1''(x(t) - bt)}{p_2''(x(t) + ct) - q_2''(x(t) - ct)} = \frac{(x'(t))^2 - c^2}{(x'(t))^2 - b^2} > 0 \tag{2.7}$$

Relationship (2.7) characterizes the jump of the second derivatives on the signoton front exactly as (2.3) characterizes the jump in the first derivatives on the shock front. It follows from (2.7) that

$$|x'(t)| > b \text{ or } |x'(t)| < c \tag{2.8}$$

We find from the first equation in (2.2) for the semisignoton that if the non-zero unilateral derivative is positive, then $|x'(t)| = c$, if it is negative then $|x'(t)| = b$. In the case of simple discontinuities $|x'(t)| = c$ if the unilateral derivatives are non-negative on the discontinuity, and $|x'(t)| = b$ if they are non-positive. It follows from (2.4) and

(2.8) that the shock and signoton velocities are in non-intersecting domains.

We refine the mentioned classification of discontinuities. We let $\alpha_+ (\alpha_-)$ denote the shock for which $x' > 0$ $(x' < 0)$, $\beta_+ (\beta_-)$ the signoton for which $x' > b$ $(x' < -b)$, β_0 the signoton for which $|x'| < c$. We call the signotons β_+, β_- fast, and the signoton β_0 slow. Analogously $\gamma_+ (\gamma_-)$ is a semisignoton whose velocity is $b\ (-b)$, and $\gamma^+ (\gamma^-)$ is a semisignoton whose velocity is $c\ (-c)$. If necessary, we shall include the subscript zero in the notation for the semisignoton to indicate from which side of its front the derivative equals zero. Namely, $\gamma_0^{\pm} \gamma_{\pm}^0$ are slow and fast semisignotons for which the derivative equals zero from the right of the front, and ${}_0\gamma^{\pm}$, ${}^0\gamma_{\pm}$ are slow and fast semisignotons for which the derivative vanishes from the left of the front.

We note that conditions (2.2) can be written in the form

$$\begin{aligned} 2bp_1 (x (t) + bt) = (b + c)\, p_2 (x (t) + ct) + \\ (b - c)\, q_2 (x (t) - ct) \\ 2bq_1 (x (t) - bt) = (b - c)\, p_2 (x (t) - ct) + \\ (b + c)\, q_2 (x (t) - ct) \end{aligned} \tag{2.9}$$

The passage from (2.2) to (2.9) also denotes the integrability of the Hugoniot conditions. The Hugoniot conditions will later be used in the form (2.9).

3. Local solutions and their diagrams. The local Cauchy problem.

We consider piecewise-smooth solutions of (0.1) for which the kind of discontinuity can change only a finite number of times on the line of discontinuity. The assumptions made regarding the discontinuities correspond to the most prevalent kind of solutions of (0.1). However, even these solutions are fairly complex in structure. Local solutions have the simplest structure, where the solution "in the large" is a set of local solutions. We call a piecewise-smooth solution of (0.1) local in the semicircle $x_0 - \varepsilon_1 < x < x_0 + \varepsilon_1$, $t_0 \leqslant t \leqslant t_0 + \varepsilon_2$ of the point (x_0, t_0) if all the lines of discontinuity in this semicircle emerge from the point (x_0, t_0), within its limits they do not intersect for $t_0 < t$, the kind of discontinuity does not change on it for $t_0 < t$, and all lines of discontinuity emerge on the line $t = t_0 + \varepsilon_2$.

Let $u (x, t)$ be a local solution of (0.1). We fix t and moving in the direction of increasing x we write the kinds of discontinuities of $u (x, t)$ successively, excluding the simple discontinuities. For instance, the sequence $\beta_-, \alpha_-, \alpha_+, \beta_-$ means that as we more from left to right along x there are a fast signoton, then two shocks and after them still another fast signoton. It is here possible that u still has simple discontinuities that are not noted in the sequence. Such a sequence of the kinds of discontinuities of the local solution is independent of t and is called a diagram of a local solution.

Understandably, not every sequence of the kinds of discontinuities is a diagram of a certain solution. For instance, because of constraints on the velocity of the motion of discontinuities (2.4) and (2.8), the diagrams $\alpha_-, \beta_-, \ldots; \alpha_+, \alpha_-, \ldots; \ldots \beta_0, \alpha_-, \ldots$, etc., are impossible. Moreover, it can be shown that becuase of the structure of (0.1) the Hugoniot conditions and the condition that the mechanical energy should not grow are not realizable by sequences of the kinds of discontinuities of the following kind:

$$\begin{gathered} \beta_-, \beta_-, \ldots; \beta_-, \gamma_-, \ldots; \alpha_-, \alpha_-, \ldots; \ldots, \gamma^-, \beta_0, \ldots; \ldots, \beta^0, \gamma^+, \ldots \\ ; \ldots, \gamma_0^-, \quad {}_0\gamma^+, \ldots; \ldots, \alpha_+, \alpha_+ \ldots; \ldots, \gamma_+, \beta_+; \ldots \beta_+, \beta_+ \end{gathered}$$

Here we have in mind two diagrams $\beta_-, \gamma_-^0, \ldots; \beta_-, {}^0\gamma_-, \ldots$, as the diagram β_-, γ_-, and both these diagrams are impossible.

If the solution in the semicircle is monotonic in x, then we denote the diagram of such a solution by O. The diagram indicates the number of sections of distinct monotonicity of the local solution. If there is a shock or a semisignoton with a refined index zero in the diagram, then the kind of monotonicity of the solution between the discontinuities can be repoduced from such a diagram.

All possible kinds of local solutions are presented in Fig.1 for fixed t. The number above the arrow in Fig.1 indicates the number of different variations in the behaviour of the solution after the arrow. For instance, if the inequality $u_x (-bt + 0, t) > 0$ is satisfied for the solution, then 42 continuations are possible in the domain $x > -bt$. These continuations consist of 21 kinds of solutions that increase monotonically in the interval $(-bt, -ct)$ and 21 kinds of solutions having a shock in this interval, etc. As follows from Fig.1, the local solutions have 189 allowable (realizable) different diagrams.

We shall call a diagram stable if any local solution sufficiently close to the given solution in the metric

$$C \left[x_0 - \frac{\varepsilon_1}{2} \leqslant x \leqslant x_0 + \frac{\varepsilon_1}{2},\, t_0 \leqslant t \leqslant t_0 + \frac{\varepsilon_2}{2} \right]$$

has the same diagram. Diagrams that do not contain semisignotons are stable. The number of such diagrams are 36 (see Fig.1). However, there are also stable diagrams that contain semisignotons. An example is the diagram $\alpha_-, {}_0\gamma^-, \gamma_0^+, \alpha_+$. A graph of $u (x, t)$ with such a digram for fixed t is displayed in Fig.2. In all there are 52 distinct stable diagrams.

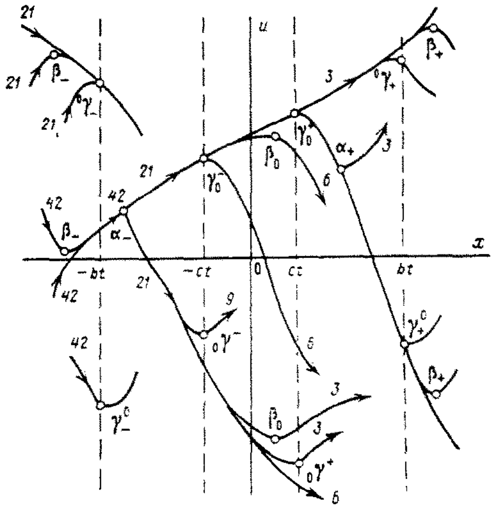

Fig.1

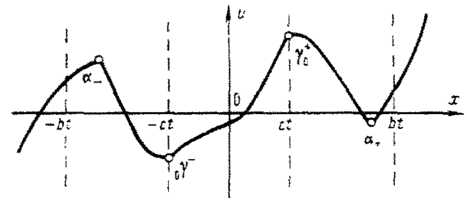

Fig.2

Only 10 diagrams correspond to piecewise-linear local solutions of (0.1) that describe the solution about dissipation of the discontinuity. In the case of (0.1), solutions of the problem about the dissociation of a discontinuity do not therefore describe the qualitative manifold of solutions of this equation at all.

The local Cauchy problem for equation (0.1) in a semineighbourhood of the point (0,0) is to seek the local solution $u(x, t)$ of (0.1) that satisfies the initial conditions

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = v_0(x) \tag{3.1}$$

Examples of existence and uniqueness theorems for the solution of a local Cauchy problem are presented in /11/.

4. The occurrence of discontinuities of the solution of (0.1). Let $u(x, t)$ be a solution of (0.1) of sufficiently high smoothness in the domain $-\varepsilon < x < \varepsilon, -\varepsilon < t < 0$. Therefore, u_x retains its sign in this domain. To be specific, we will assume that $u_x > 0$, $u_x(0, 0) = 0$, and therefore, for $t < 0$ the function u satisfies the equation

$$u_{tt} - c^2 u_{xx} = 0 \tag{4.1}$$

We will assume that u_x changes sign as x changes for the continuation u as a solution of (4.1) for $t > 0$. This means that discontinuities occur at the point (0,0) for the solution of (0.1) that agrees with u for $t < 0$.

We will first examine a typical process (general situation) for the occurrence of discontinuities ($u(x, t)$ is a versal deformation with parameter t /12/). For $t < 0$ the solution $u(x, t)$ of (0.1) has the form $u(x, t) = p(x + ct) + q(x - ct)$. The functions p, q are assumed to be sufficiently smooth, and $p(0) = q(0) = 0$. We will limit ourselves to expansions of the functions p, q at zero by the Taylor formula to the cubes of the arguments inclusive

$$u(x, t) = \alpha(x + ct)^3 + \beta(x + ct)^2 + \gamma(x + ct) + \delta(x - ct)^3 + \lambda(x - ct)^2 + \mu(x - ct) \tag{4.2}$$

Since a critical point occurs in the solution at $t = 0$, then $u(x, 0) = \sigma x^3, \sigma > 0$, i.e., $\gamma + \mu = 0, \beta + \lambda = 0, \alpha + \delta = \sigma$. Without loss of generality, we can consider that $u(x, t)$ has the following form for $t < 0$:

$$u(x, t) = \alpha(x - ct)^3 + \beta(x + ct)^2 + \delta(x - ct)^3 - \beta(x - ct)^2 \tag{4.3}$$

The occurrence of a discontinuity at the point (0,0) is ensured by the inequalities $\alpha + \beta > 0, \beta < 0$. By solving the equation $u_x(x, t) = 0$, we find the fronts of the fast signotons $x = x_+(t), x = x_-(t)$ from (4.3). Furthermore, the solution for $x_-(t) \leqslant x \leqslant x_+(t)$ is found from the relationship (2.5). The time of existence of the solution with diagram β_-, β_+ is determined from the conditions $x_-'(t) < -b, x_+'(t) > b$. It can be shown that if $|\alpha| \to \infty, |\delta| \to \infty, 0 < c_1 \leqslant |\alpha - \delta| / |\alpha + \delta| \leqslant c_2$, then the time of existence of a solution with diagram β_-, β_+ tends to zero.

Thus, as a rule, for a smooth solution for $t < 0$ two fast signotons occur at the time $t = 0$ which can undergo further transformation after a certain time. If the time of existence of these fast signotons is quite small, then it is natural to consider the passage at once from a smooth solution for $t < 0$ to the next stage in the behaviour of the discontinuities by bypassing the stage associated with the fast signotons. This is why it is worth investigating schemes for the occurrence of discontinuities not of general location.

We will present some examples of processes for the occurrence of discontinuities that are different from the passage $0 \to \beta_-, \beta_+$. For $t<0$ let the solution $u(x, t)$ have the form

$$u(x,t) = \begin{cases} p_l(x+ct) + q_l(x-ct), & x \leqslant ct \\ p_l(x+ct) + q_r(x-ct), & ct \leqslant x \leqslant -ct \\ p_r(x+ct) + q_r(x-ct), & x \geqslant -ct \end{cases} \tag{4.4}$$

$$p_l(\eta) = -A|\eta|^{2n}, \quad q_l(\eta) = -B|\eta|^{2n}, \quad p_r(\xi) = C|\xi|^{2n}$$
$$q_r(\xi) = D|\xi|^{2n}, \quad n \geqslant 1$$

The condition $u_x \geqslant 0$ for $t<0$ is ensured by the inequalities $A>0$, $A+B>0$, $D>0$, $C+D>0$. We note that if the numbers A, B, C, D are positive, then the solution remains monotonically increasing even for $t>0$.

We shall seek fast signotons for the solution for $t>0$. We obtain for their fronts

$$x_-(t) = -\frac{(A^\lambda - B^\lambda)ct}{A^\lambda + B^\lambda} = k_- t \tag{4.5}$$

$$x_+(t) = \frac{(D^\lambda - C^\lambda)ct}{D^\lambda + C^\lambda} = k_+ t; \quad \lambda = \frac{1}{2n-1}$$

If $k_- < -b$, $k_+ > b$, then (4.5) indeed yields the fronts of fast signotons even for $t>0$. If $k_- > -b$ or $k_+ < b$, then for $t>0$ the solution cannot have the diagram β_-, β_+.

We note that for $t>0$ the solution can be sought in the form of a homogeneous function of degree $2n$. Consequently, the fronts of the discontinuities are straight lines. For simplicity we limit ourselves to the case $n=1$. For $n>1$ the investigations are performed analogously. To be specific, let $k_- > -b$, $k_+ > b$, and therefore $C<0$. Hence, for $t>0$ there is a fast signoton in the diagram. We shall seek the solution with diagram α_-, β_+ for $t>0$. From (2.5) and (2.9), for $x=\alpha t$ the shock front, we obtain the equation

$$(b+c)A(\alpha+c)^2 + (b-c)B(\alpha-c)^2 = -\frac{4CDc^2(\alpha+b)^2}{c(D-C)-b(D-C)} \tag{4.6}$$

Setting $B=0$ in (4.6) and taking into account that $A>0, C<0, D>0$, we obtain the existence of a solution of (4.6) such that $-b<\alpha<-c$. The existence of a solution with diagram α_-, β_+ is thereby proved.

We will examine another solution for $t>0$ that has the digram β_0, β_+. For the front $x=\beta_0 t$ of a slow signoton, we obtain the equation

$$B(c-\beta_0)(c(D-C)+b(D+C)) = -2CDc(b-\beta_0) \tag{4.7}$$

Having been given the numbers C, D, β_0, where $|\beta_0|<c$, we find the number B from (4.7). For the passage $0 \to \beta_0, \beta_+$ the appearance of singularities in the solution for $t>0$ is possible that appear in the power-law growth of the second derivatives on one of the lines $x=\pm ct$. Although the passage $0 \to \beta_0, \beta_+$ is not typical, as remarked above, and can vanish for $t<0$ for small changes in the solution, the appearance of this singularity affects values of the second derivative that are large in mangitude. The occurrence of a power-law singularity for a solution that is smooth for $t<0$ is of interest in connection with the fact that the corresponding derivatives are discontinuous in shocks and signotons, but the solution has unilateral derivatives of fairly high orders.

We will examine in greater detail the origination of power-law singularities in the solution during the passage $0 \to \beta_0, \gamma_+^0$, which is the passage to the limit $0 \to \beta_0, \beta_+$ and is technically simpler to investigate. For $t<0$ let the solution u be represented in the form (4.4) where the functions p_l, q_l, p_r, q_r are such that

$$p_l(0) = q_l(0) = p_r(0) = q_r(0) = p_l'(0) = q_l'(0) = p_r'(0) = q_r'(0) = 0$$

Other conditions on these functions will be presented below. Since the solution contains a semisignoton γ_+^0 for $t>0$, then

$$q_r(\xi) = -\frac{b-c}{b+c}\, p_r\left(\frac{b+c}{b-c}\,\xi\right) \tag{4.8}$$

The non-decrease of u in x for $x \geqslant bt > 0$ follows from the inequality $p_r''(\xi) < 0$ for $\xi>0$. Since $p_r'(0)=0$, it follows from the inequality $p_r''(\xi)<0$ that $p_r'(\xi)<0$ for $\xi>0$. The condition of monotonic growth of u for $t<0$ results from the inequalities $p_r''(\xi)<0$ for $\xi>0$, $p_l'(\eta)>0$ for $\eta<0$, $p_l'(\eta_1)+q_l'(\eta_2)>0$ for $\eta_1 \leqslant \eta_2 < 0$. For $t>0$ we shall seek a solution with diagram β_0, γ_+^0. Then $u(x,t)$ is given by the formulas

$$u(x,t) = \begin{cases} p_l(x+ct) + q_l(x-ct), & x \leqslant -ct \\ p(x+ct) + q_l(x-ct), & -ct \leqslant x \leqslant \beta(t) \\ p_*(x+bt) + q(x-bt), & \beta(t) \leqslant x \leqslant bt \\ p_r(x+ct) + q_r(x-ct), & bt \leqslant x \end{cases}$$

The condition for merger on the line $x = bt$ determines the function

$$p_*(\xi) = \frac{2c}{b+c} p_r\left(\frac{b+c}{2b}\xi\right) \tag{4.9}$$

We take the front of the slow signoton in the form $\beta(t) = -ct + kt^M$, $k > 0$, $M > 1$. We express the function p, q_l, q in terms of p_* from (2.5) and the form of $\beta(t)$

$$\begin{aligned} q'(-(b+c)t + kt^M) &= -p_*'((b-c)t + kt^M) \\ q_l'(-2ct + kt^M) &= -bc^{-1}p_*'((b-c)t + kt^M) \\ p'(kt^M) &= bc^{-1}p_*'((b-c)t + kt^M) \end{aligned} \tag{4.10}$$

Since $p_*' < 0$ and $q_l' > 0$, then u is a monotonically increasing function for $-ct \leqslant x \leqslant \beta(t)$ and monotonically decreasing for $\beta(t) \leqslant x \leqslant bt$.

Let the function $p_r(\xi)$ be infinitely differentiable for $\xi > 0$ and $M > 1$ an integer. Then it follows from (4.9) and (4.10) that $p_l(\xi)$, $q(\eta)$, $q_l(\eta)$ are infinitely differentiable functions for $\xi > 0, \eta < 0$. We set $p_r''(\xi) = A\xi^{N-2} + o(\xi^{N-2})$. In this case, we have from (4.9) that $p_l''(\xi) = B\xi^{N-2} + o(\xi^{N-2})$. We find from the last inequality in (4.10)

$$p''(kt^M) = \frac{bB}{ckM}(b-c)^{N-2}t^{N-M-1} - o(t^{N-M-1}) \tag{4.11}$$

Equation (4.11) shows that the second derivatives of u can have a power-law singularity on the line $x = -ct$. For instance, if $M = 2$, $N = 2$, then $u_{xx}(x + ct) \sim C(x + ct)^{-1/2}$ for $x + ct > 0$. Thus, by taking the function $p_l(\eta)$ such that $p_l'(\eta) > 0$ for $\eta < 0$, and $p_r(\xi)$ such that $p_r''(\xi) < 0$ for $\xi > 0$, we find u with the diagram β_0, γ_+^0 from (4.8), (4.9), (4.10).

5. Bifurcations of discontinuities.

Let $x = x(t)$ be the front of a discontinuity which is a smooth curve for $t_1 < t < t_2$, on which the kind of discontinuity is conserved. We shall say that a discontinuity bifurcation occurs at the point $(x(t_2), t_2)$ if branching of the discontinuity occurs for $t > t_2$ at this point or the kind of discontinuity changes during passage through this point. It is assumed that there are no other discontinuities except $x = x(t)$ in a certain semineighbourhood of the point $(x(t_2), t_2)$, $t \leqslant t_2$.

We first consider the bifurcation of the fast signotons. For $t < 0$ let the solution have the diagram β_- and a signoton whose front $x = \beta(t)$ is a local maximum, where $\beta(t)$ is assumed to be a sufficiently smooth function and $\beta'(t) = -b - 2dt + O(t^2)$, $d > 0$. Then the solution allows of the representation

$$u(x, t) = \begin{cases} p_l(x - ct) + q_l(x - ct): & x \leqslant \beta(t) \\ p_r(x - bt) + q_r(x - bt). & x \geqslant \beta(t) \end{cases}$$

The functions p_i, q_i are assumed to be sufficiently smooth in a certain neighbourhood of zero. Without loss of generality, it can be assumed that

$$p_l(0) = q_l(0) = p_r(0) = q_r(0) = p_l'(0) = q_l'(0) = p_r'(0) = q_r'(0) = 0$$

A typical case of bifurcation corresponds to the inequality

$$p_l''(0) - q_l''(0) < 0 \tag{5.1}$$

It follows from inequality (5.1) and the relationship (2.5) that $p_l''(0) \neq 0$, $q_l''(0) \neq 0$, and we find from the condition $u_x > 0$ for $t < 0$ that $p_l''(0) < 0$, $q_l''(0) > 0$. For $t > 0$ we will seek a solution with diagram α_- in the form

$$u(x, t) = \begin{cases} p_l(x - ct) + q_l(x - ct), & x \leqslant \alpha(t) \\ p_r(x - bt) + q(x - bt), & \alpha(t) \leqslant x \leqslant bt \\ p_r(x - bt) + q_r(x - bt), & x \geqslant bt \end{cases}$$

The unknown functions $\alpha(t)$, $q(\eta)$ are determined from (2.9). Namely, from the first equation in (2.9) we find $\alpha(t)$ by the theorem on implicit functions. The second equation in (2.9) determines $q(\eta)$.

We set $\alpha(t) = -bt + k(t)t^2$. In order for the shock velocity to be incident in the appropriate zone, it is sufficient that $k(0) > 0$. We set $p_l(x) = -x^2 + Px^3 + O(x^4)$ so that $p_l''(x) = -2 + 6Px + O(x^2)$. We find q_l, p_r, q_r from the relationship (2.5). In particular

$$q_l(x) = \frac{b-c}{b+c}x^2 - \frac{x^3}{3}\left(\frac{4dc}{(b-c)^3} - 3P\left(\frac{b-c}{b+c}\right)^2\right) + O(x^4)$$

$$p_r(\xi) = -\frac{4c(b-c)}{3\sqrt{d}\,b}\xi^{3/2} + O(\xi^2)$$

Substituting the expressions for p_l, q_l, p_r, α into the first equation of (2.9), we find that

$$(b-c)k^{3/2}(t) = {}^1/_2(b-c)d^{1/2} + O(t) \tag{5.2}$$

If $b \neq c$, we obtain $k(0) = 2^{-1/3}d$ from (5.2). Therefore, $k(0) > 0$. The monotonicity conditions for u are verified directly for $t > 0$. We note that the ratio of the signoton and shock front accelerations at the bifurcation point equal $2^{2/3}$ and therefore is independent of the quantity a in (0.1), although this result is only valid for $a \neq 0$.

We once again examine the question of the bifurcation of a shock. For $t < 0$ let the shock front have the form

$$\alpha(t) = -ct + kt^2 + O(t^3),\ k > 0,\ O(t^3) = t^3 r(t)$$

($r(t)$ is a fairly smooth function), and the solution allows of the representation

$$u(x,t) = \begin{cases} p_l(x+ct) + q_l(x-ct), & x \leqslant \alpha(t) \\ p_r(x+bt) + q_r(x+bt), & x \geqslant \alpha(t) \end{cases}$$

We set

$$p_l(x) = Ax + Bx^2 + O(x^3).\ q_l = Cx + Dx^2 + O(x^3)$$

We find from the relationships (2.9) and the form of $\alpha(t)$

$$p_r(x) = -C\frac{c}{b} + ((b+c)(A+C)k + (b-c)4Dc^2)\frac{x^2}{2b(b-c)} + O(x^3)$$

$$q_r(x) = C\frac{c}{b} + ((b-c)(A+C)k + (b-c)4Dc^2)\frac{x^2}{2b(b+c)} + O(x^3)$$

For simplicity we will limit ourselves to the case when $A + C > 0$. We assume that $p_r''(0) + q_r''(0) < 0$. This inequality ensures that there are no collisions of discontinuities and is equivalent to the relationship

$$\frac{(2-a)(A+C)k}{4a(1-a)} < -D \tag{5.3}$$

The occurrence of the signoton β_+ for $t > 0$ can be determined from the functions p_r, q_r. If

$$-D < D_*,\quad D_* = \frac{(b+c)(A+C)k}{4(b-c)c^2} \tag{5.4}$$

then for $t = 0$ a fast signoton β_+ occurs, if

$$-D > D_* \tag{5.5}$$

then no fast signoton β_+ occurs for sufficiently small $t > 0$.

We note that the left side of (5.3) is always less than the right side of (5.4), i.e., a solution with a fast signoton β_+ is generally possible. In this case the typical diagram for $t > 0$ has the form $\beta_0, \alpha_+, \beta_+$ and the investigation of such a solution is always awkward.

Let the inequality (5.5) be satisfied. For $t > 0$ we shall seek the solution with diagram β_0. For the slow signoton front $x = \beta(t)$ we obtain from (2.5) the equation

$$p_r'(\beta(t) + bt)\, b = -cq_l'(\beta(t) - ct)$$

from which we have

$$\beta'(0) = -\frac{b(A+C)k + (b-c)2Dc^2}{(A-C)k + (b-c)2Dc}$$

It follows from condition (5.5) that $|\beta'(0)| < c$. The passage $\alpha_- \to \beta_0$ considered describes the process of shock disappearance.

We will now consider special cases for the occurence of discontinuities for smooth solutions as examined in Sect.4. In the first, the smooth solution for $t < 0$ generates a shock for $t > 0$, and the second a slow signoton. The results in Sect.5 are the foundation for the examination of these passages. Namely, if a fast signoton occurs for $t = 0$, then as follows from Sect.5, the transformation into a shock is typical for it. Therefore, the first of the special solutions in Sect.4 approximates the solution of the general situation with a small existence time for the fast signoton. The second solution in Sect.4 characterizes the solution of the general situation with small existence times for both the fast signoton and the shock being formed from it.

6. Local solutions of equation (0.3).

The concept of a local solution of (0.3) is analogous to the corresponding concept for (0.1) (see Sect.3). Namely, a rectangle Π: $x_0 - \varepsilon_1 < x < x_0 + \varepsilon_1$, $t_0 \leqslant t \leqslant t_0 + \varepsilon_2$ is considered, as well as a fan of smooth curves Γ_i: $x = x_i(t)$, $t_0 \leqslant t < t_0 + \varepsilon_2$, $i = 1, \ldots, N$, starting from the points x_0, t_0; $x_0 = x_i(t_0)$ and do not intersect in Π for $t > t_0$. We denote by Π_i, $i = 0, 1, \ldots, N$ an open connected domain in Π that is bounded by the adjacent sides Γ_i, Γ_{i+1} and the sides of Π.

The function $u(x,t)$ defined in Π and satisfying the following conditions is called a local solution of (0.3). In the domains Π_i, $i = 0, \ldots, N$ the function $u(x,t)$ is twice continuously differentiable, monotonic in x, where the nature of the monotonicity is independent of t and satisfies (0.3). The lines Γ_i are called lines (fronts) of discontinuity of the solution. To complete the definition of the concept of a local solution it remains to describe the behaviour of the function $u(x, t)$ during passage through the front of the discontinuity.

Four kinds of discontinuities $\alpha, \beta, \gamma, \delta$ were considered in Sect.2. These kinds of discontinuities were sufficient to describe the evolution of smooth initial functions of the Cauchy problem for the equation (0.1). These four kinds of discontinuities are also conserved for (0.3), where the Hugoniot conditions (1.2) and the condition of local non-growth of the mechanical energy (1.3) should be satisfied for the α-discontinuity. However, the above-mentioned four kinds of discontinuities turn out to be insufficient for a solution of the Cauchy problem to exist for (0.3) with smooth initial functions.

To ensure the existence of a solution of the problem under consideration, one other kind of discontinuity must be introduced, namely, a λ-discontinuity (a discontinuity in the continuous displacement, spall). Its front is fixed, i.e., $x(t) = x_0$. Taking into account that x is a Lagrange coordinate of the particle, we find that after the formation of the λ-discontinuity, the whole system dissociates into two inexchangeable particles of the system. Furthermore, it is assumed that $u^+ > u^-$ and $(u_x)^\pm \geqslant 0$.

We note that in the case of (0.3) the fronts of the γ- and δ-discontinuities emerging from the point $x = 0, t = 0$ have the form $x = \pm\sqrt{2}t$, $x = 0$. If $x(t)$ is the slow signoton front emerging from the point $x = 0, t = 0$, then $x(t) = 0$, i.e., the front is fixed for the slow signoton.

We consider the condition on the α-, β-discontinuities in a form taking account of the specific features of Eq.(0.3). The corresponding conditions used in constructing solutions of specific problems are in this form. We let $u^1(x, t)$ ($u^2(x, t)$) denote the solution $u(x, t)$ of (0.3) in the domain where $u_x(x, t) \leqslant 0$ ($u_x(x, t) \geqslant 0$). Then considering $x_0 = 0$, $t_0 = 0$, we have

$$u^1(x, t) = p(x + \sqrt{2}t) + q(x - \sqrt{2}t) - Ax^2/4 \tag{6.1}$$

$$u^2(x, t) = a(x) + b(x)t + At^2/2$$

From the relationships (6.1) and the Hugoniot conditions (1.2) we find

$$2\sqrt{2}p(x(t) \pm \sqrt{2}t) = \sqrt{2}(a(x(t)) + tb(x(t))) \pm B(x(t)) + A(\sqrt{2}/4)(x(t) \pm \sqrt{2}t)^2 \tag{6.2}$$

where $B'(\lambda) = b(\lambda)$, and $x(t)$ is the front of the α-discontinuity (shock).

If $x(t)$ is the front of a β-discontinuity (fast signoton), then by using (6.1) the conditions on the front can be written in the form

$$a'(x(t)) + tb'(x(t)) = 0, \quad p'(x(t) + \sqrt{2}t) + q'(x(t) - \sqrt{2}t) = Ax(t)/2 \tag{6.3}$$

$$b(x(t)) = -At + \sqrt{2}p'(x(t) + \sqrt{2}t) - \sqrt{2}q'(x(t) - \sqrt{2}t)$$

We note that the inequality

$$|\alpha'(t)| \leqslant \sqrt{2} \tag{6.4}$$

is satisfied for the shock front velocity $\alpha'(t)$ while we have $|\beta'(t)| \geqslant \sqrt{2}$ for the velocity $\beta'(t)$ of the fast signoton front.

Using the general properties of the solutions of (0.3), we will proceed to the solution of specific problems for this equation.

7. Occurence of λ-discontinuities (discontinuities in displacement). We consider the simplest problem for (0.3) for $A = 0$ on the collision of two unformed systems in which the spalling phenomenon occurs. Let $-l_1 < x < l_2$, $l_1, l_2 > 0$. A solution of (0.3) is sought that satisfies the conditions

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = v_0(x),$$

$$u_x(-l_1, t) \geqslant 0, \quad u_x(l_2, t) \geqslant 0$$

$$u_0(x) = 0, \quad v_0(x) = \begin{cases} V > 0, & -l_1 \leqslant x < 0 \\ 0, & 0 \leqslant x \leqslant l_2 \end{cases}$$

The solution $u(x, t)$ for $t < 0$ is given by the formulas

$$u(x, t) = \begin{cases} Vt, & -l_1 \leqslant x \leqslant 0 \\ 0, & 0 \leqslant x \leqslant l_2 \end{cases}$$

i.e., for $t < 0$ the left system $-l_1 \leqslant x < 0$ moves to the right of a constant velocity V, while the right system $0 \leqslant x \leqslant l_2$ is fixed.

To be specific, let $l_1 \leqslant l_2$. We present the solution of the formulated problem for $t > 0$.

If $0 \leqslant \sqrt{2}t \leqslant l_1$, then

$$u(x,t) = \begin{cases} Vt, & -l_1 \leqslant x \leqslant -\sqrt{2}t \\ V(\sqrt{2}t - x)/2\sqrt{2}, & -\sqrt{2}t \leqslant x \leqslant \sqrt{2}t \\ 0, & \sqrt{2}t \leqslant x \leqslant l_2 \end{cases}$$

If $l_1 \leqslant \sqrt{2}t \leqslant l_2$, then

$$u(x,t) = \begin{cases} Vl_1/\sqrt{2}, & -l_1 \leqslant x \leqslant -2l_1 + \sqrt{2}t \\ V(\sqrt{2}t - x)/2\sqrt{2}, & -2l_1 + \sqrt{2}t \leqslant x \leqslant \sqrt{2}t \\ 0, & \sqrt{2}t \leqslant x \leqslant l_2 \end{cases}$$

If $l_2 \leqslant \sqrt{2}t \leqslant l_1 + l_2$, the solution has the form

$$u(x,t) = \begin{cases} Vl_1/\sqrt{2}, & -l_1 \leqslant x \leqslant -2l_1 + \sqrt{2}t \\ V(\sqrt{2}t - x)/2\sqrt{2}, & -2l_1 + \sqrt{2}t \leqslant x \leqslant 2l_2 - \sqrt{2}t \\ V(t - l_2/\sqrt{2}), & 2l_2 - \sqrt{2}t \leqslant x \leqslant l_2 \end{cases}$$

Thus, up to the time $t_1 = (l_1 + l_2)\sqrt{2}$ the inequality $u_x \leqslant 0$ is satisfied, and therefore, the problem is solved within the framework of the linear formulation.

For $t > (l_1 + l_2)/\sqrt{2}$ the solution has the form

$$u(x,t) = \begin{cases} Vl_1/\sqrt{2}, & -l_1 \leqslant x \leqslant l_2 - l_1 \\ V(t - l_2/\sqrt{2}), & l_2 - l_1 \leqslant x \leqslant l_2 \end{cases}$$

i.e., at the point $x = l_2 - l_1$ a fixed discontinuity again originates.

Therefore, the impact of the left system of length l_1 moving to the right at a velocity V in a fixed right system, results after a certain time in the fact that a piece of length l_1 is torn off from the right system, which will move to the right with velocity V while the remaining particles will be fixed.

8. Occurence of a shock from a static initial state. Let us consider a Cauchy problem for (0.3) with $A = 0$ and the initial conditions

$$u(x, 0) = u_0(x), \ u_t(x, 0) = 0$$

where $u_0(x)$ is a continuous function, $u_0'(x) > 0$ for $x < 0$, $u_0'(x) < 0$ for $x > 0$. For $t > 0$ we seek the solution in the form

$$u(x,t) = \begin{cases} u_0(x), & x \leqslant \alpha(t) \\ p(x - \sqrt{2}t) - q(x - \sqrt{2}t), & \alpha(t) \leqslant x \leqslant \sqrt{2}t \\ (u_0(x + \sqrt{2}t) + u_0(x - \sqrt{2}t))/2, & x \geqslant \sqrt{2}t \end{cases}$$

where $x = \alpha(t)$ is the shock front, and $p(\xi) = u_0(\xi)/2$, $\xi \geqslant 0$.

The function $\alpha(t)$ is found from the first equation in (6.2): $u_0(\alpha(t) + \sqrt{2}t) = u_0(\alpha(t))$. Furthermore, knowing $\alpha(t)$, we find $q(\eta)$ for $\eta \leqslant 0$ from (6.2) with the lower minus sign. Then the monotonicity condition $u_x \leqslant 0$ should be confirmed for $\alpha(t) \leqslant x \leqslant \sqrt{2}t$, as should inequality (6.4).

Let $u_0(x)$ be an even function; then $\alpha(t) = -\sqrt{2}t/2$ and $q(\eta) = u_0(\eta/3)/2$. The appropriate conditions of monotonicity for sufficiently small $t > 0$ are confirmed directly. Therefore, the solution actually has the diagram α_-.

We note that more complex structures of the solution are indeed possible in the problem under consideration. For instance, if $u_0(x) = -|x|^{1/2}$, then for $t > 0$ the solution has the diagram $\alpha_-, \alpha_+, {}^0\gamma_+$. In this case the solution is a homogeneous function of degree $1/2$, and the shock fronts are lines

$$x = \alpha_- t, \quad x = \alpha_+ t, \quad \alpha_- = -\frac{3\sqrt{2} - 19}{49}, \quad \alpha_+ = \frac{4 - \sqrt{2}}{7}$$

In the problem under consideration the solution in the case of a smooth function $u_0(x)$ generally has the diagram α_- only in a small time interval.

9. Collision of a rarefield system with a rigid wall ($A = 0$). Let $x \geqslant 0$ and the system be located to the right of the wall $x = 0$ for $t < 0$, i.e., $u_x(x, t) \geqslant 0$, $x + u(x, t) \geqslant 0$. Furthermore, let

$$u(x, 0) = u_0(x), \ u_t(x, 0) = -v(x), \ u_0'(x) \geqslant 0, \ v(0) > 0,$$
$$u_0(0) = 0,$$

i.e., near $x = 0$ the system moves to the wall $x = 0$ for $t < 0$.

For small $t > 0$ near $x = 0$ the solution has the form

$$u(x,t) = \begin{cases} p(x+\sqrt{2}t) - p(\sqrt{2}t - x), & 0 \leqslant x \leqslant \alpha(t) \\ u_0(x) - tv(x), & x \geqslant \alpha(t) \end{cases}$$

Therefore, a compression zone abuts on the wall, which is terminated by a shock whose front is $x = \alpha(t)$. From (6.2) we find the equations for $p(\xi)$, $\alpha(t)$

$$\pm 2\sqrt{2}p(\sqrt{2}t \pm \alpha(t)) = \sqrt{2}(u_0(\alpha(t)) - tv(\alpha(t))) - V(\alpha(t)) \qquad (9.1)$$

where $V'(\xi) = v(\xi)$, $V(0) = 0$. From (9.1) we obtain $\alpha'(0)$, $p'(0)$

$$\alpha'(0) = \frac{2v_0}{u_0' + \sqrt{(u_0')^2 + 2v_0^2}} \qquad (9.2)$$

$$p'(0) = -\frac{1}{4}(\sqrt{(u_0')^2 + 2v_0^2} - u_0'),\ u_0' = u_0'(0),\ v_0 = v(0)$$

If $u_0(x) = kx$, $v(x) = v_0$, then (9.2) yields the solution for all $t > 0$. Namely, $\alpha(t) = \alpha'(0)t$, $p(\xi) = p'(0)\xi$.

We trace the whole collision process in this case (for $0 \leqslant x \leqslant l$). For $0 \leqslant t \leqslant l/\alpha$

$$u(x,t) = \begin{cases} 2px, & 0 \leqslant x \leqslant \alpha t \\ kx - v_0 t, & \alpha t \leqslant x \leqslant l \end{cases}$$

If $l/\alpha \leqslant t \leqslant l/\alpha + l/\sqrt{2}$, then

$$u(x,t) = \begin{cases} 2px, & 0 \leqslant x \leqslant l - \sqrt{2}(t - l/\alpha) \\ 2p(l - \sqrt{2}t + l\sqrt{2}/\alpha), & l - \sqrt{2}(t - l/\alpha) \leqslant x \leqslant l \end{cases}$$

Finally, if $t > l/\alpha + l/\sqrt{2}$, then $u(x,t) = 2p(l + l\sqrt{2}/\alpha - \sqrt{2}t)$ when $0 \leqslant x \leqslant l$. Therefore for $t > l/\alpha + l/\sqrt{2}$ the system becomes compact ($u_x = 0$) and moves to the right at a constant velocity

$$v_1 = \sqrt{2}\frac{v_0^2}{k - \sqrt{k^2 - 2v_0^2}} < v_0$$

Thus, a uniformly rarefied system ($u_x = k > 0$), moving to the left at a constant velocity v_0 because of a collision with a rigid wall, will recoil from it after a certain time and will move to the right with constant velocity v_1, $v_1 < v_0$, as a compact system ($u_x = 0$).

10. Reflection of a compression wave from a free edge.

The following problem is considered for (0.3) with $A = 0$. For $t < 0$ let the solution of (0.3) have the form

$$u(x,t) = \begin{cases} 0, & 0 \leqslant x \leqslant -\sqrt{2}t, \\ f(x - \sqrt{2}t), & -\sqrt{2}t \leqslant x, \end{cases} \quad f'(\xi) \leqslant 0,\ \xi \geqslant 0,\ f(0) = 0$$

The problem is to find the solution for $t > 0$ under the condition $u_x(0, t) \geqslant 0$. It turns out that the form of the solution depends substantially on $f''(\xi)$. Hence, we consider different cases of the behaviour of $f''(\xi)$.

Let $f''(\xi) < 0$, then for $t \geqslant 0$ the solution has the form

$$u(x,t) = \begin{cases} f(x - \sqrt{2}t) - f(\sqrt{2}t - x), & 0 \leqslant x \leqslant \sqrt{2}t \\ f(x + \sqrt{2}t), & x \geqslant \sqrt{2}t \end{cases} \qquad (10.1)$$

Indeed, if $0 \leqslant x \leqslant \sqrt{2}t$, then

$$u_x = f'(\sqrt{2}t + x) - f'(\sqrt{2}t - x) = f''(\xi)\cdot 2x$$

Therefore, in this case the solution of (0.3) agrees with the solution of the linear problem (a compression wave reflected from a free edge remains a compression wave).

If $f'(\xi)$ is a monotonically increasing function, then the solution of (0.3) for $t \geqslant 0$ has the form

$$u(x,t) = \begin{cases} f(2x) + 2f'(2x)(\sqrt{2}t - x), & 0 \leqslant x \leqslant \sqrt{2}t \\ f(x + \sqrt{2}t), & x \geqslant \sqrt{2}t \end{cases} \qquad (10.2)$$

In this case a semisignoton ${}^0\gamma_+$ occurs at the point $x = 0$ for $t = 0$.

We note that if $f(\xi)$ has a discontinuity of the first derivative at the point ξ_0, then for $t \geqslant \xi_0/2\sqrt{2}$ a discontinuity in the displacement occurs at the point $x_0 = \xi_0/2$. If $f'(\xi) = 0$ for $\xi \geqslant \xi_1$, then for $x > \xi_1/2$ and $t > \xi_1/2\sqrt{2}$ the reflected wave propagates only a finite distance to the right.

We now consider several cases of a compression wave reflected from a free edge, when $f''(\xi)$ changes sign for $\xi = \xi_0$, $f''(\xi_0) = 0$. Let $f''(\xi) \leqslant 0$ for $\xi \leqslant \xi_0$, $f''(\xi) \geqslant 0$ for $\xi \geqslant \xi_0$. We consider the function $\xi = \xi(y)$, i.e. the middle of segements connected identical values of the function $y = f'(\xi)$, $y_2 \leqslant y \leqslant y_1 \leqslant 0$.

Let $\xi(y)$ be a monotonically decreasing function such that

$$\min \xi(y) = \xi(y_1) = \xi_1, \quad \max \xi(y) = \xi(y_2) = \xi_0, \quad 2\xi_1 > \xi_0$$

In this case the solution has the form (10.1) for $0 \leqslant \sqrt{2}t \leqslant \xi_1$. If $\xi_1 \leqslant \sqrt{2}t \leqslant \xi_0$, then

$$u(x,t) = \begin{cases} f(x+\sqrt{2}t) + f(\sqrt{2}t - x), & 0 \leqslant x \leqslant \beta(t) \\ a(x) + tb(x), & \beta(t) \leqslant x \leqslant \xi_1 \\ c(x) + td(x), & \xi_1 \leqslant x \leqslant \sqrt{2}t \\ f(x+\sqrt{2}t), & x \geqslant \sqrt{2}t \end{cases}$$

Here $x = \beta(t)$ is the front of the signoton β_-, $x = \sqrt{2}t$ is the front of the semi-signoton ${}^0\gamma_+$. The front $x = \beta(t)$ is determined from the equation

$$f'(\sqrt{2}t + \beta(t)) - f'(\sqrt{2}t - \beta(t)) = 0$$

We hence find that

$$\sqrt{2}t + \beta(t) \geqslant \xi_0, \quad \sqrt{2}t - \beta(t) \leqslant \xi_0, \quad \beta'(t) < 0$$

since $\xi(y)$ is a decreasing function. We find from the equation for $\beta(t)$ that

$$\beta'(t) = -\sqrt{2}\,\frac{f''(\sqrt{2}t - \beta(t)) - f''(\sqrt{2}t + \beta(t))}{f''(\sqrt{2}t - \beta(t)) + f''(\sqrt{2}t + \beta(t))}$$

and, therefore, $\beta'(t) < -\sqrt{2}$, i.e., $x = \beta(t)$ is actually the signoton front. The functions $c(x), d(x), a(x), b(x)$ have the form

$$c(x) = f(2x) - 2xf'(2x), \quad d(x) = 2\sqrt{2}f'(2x)$$
$$b(x) = \sqrt{2}\,(f'(\sqrt{2}t(x) + x) + f'(\sqrt{2}t(x) - x))$$
$$a(x) = f(\sqrt{2}t(x) + x) + f(\sqrt{2}t(x) - x) - t(x)\,b(x).$$

where $t = t(x)$ is a function inverse to $x = \beta(t)$.

The monotonicity condition for the function $c(x) + td(x)$ is confirmed directly, while the monotonicity of the function $a(x) + tb(x)$ follows from the fact that $\beta(t) \leqslant x \leqslant \xi_1$, $t(x)$ is a monotonically decreasing function, and therefore $t \geqslant t(x) \geqslant \xi_1/\sqrt{2}$. Thus, in this case the solution has the diagram $\beta_-, {}^0\gamma_+$ in semineighbourhood of the point $(\xi_1, \xi_1/\sqrt{2})$.

If $\xi = \xi(y)$ is a monotonically increasing function, then for $0 \leqslant t \leqslant \xi_0\sqrt{2}$ the solution is the same as in the initial stage of the preceding example. If $t \geqslant \xi_0\sqrt{2}$ then

$$u(x,t) = \begin{cases} a(x) + tb(x), & 0 \leqslant x \leqslant \beta(t) \\ f(x+\sqrt{2}t) + f(\sqrt{2}t - x), & \beta(t) \leqslant x \leqslant \sqrt{2}t \\ f(x+\sqrt{2}t), & x \geqslant \sqrt{2}t \end{cases}$$

where $x = \beta(t)$ is the front of the signoton $(\beta' > \sqrt{2})$, the functions $\beta(t)$, $a(x)$, $b(x)$ are determined by the same formulas as in the preceding example. In this case the signoton occurs at $x = 0$ and moves to the right while in the previous case it occurred at $x = \xi_1$ and moves to the left.

We examine the problem of compression wave reflection from a free edge when $f''(\xi) \geqslant 0$ for $0 \leqslant \xi \leqslant \xi_0$, $f''(\xi) \leqslant 0$ for $\xi \geqslant \xi_0$. Here for $0 \leqslant 2\sqrt{2}t \leqslant \xi_0$ the solution has the form (10.2). For $2\sqrt{2}t \geqslant \xi_0$ we seek the solution in the form

$$u(x,t) = \begin{cases} f(2x) - 2xf'(2x)(\sqrt{2}t - x), & 0 \leqslant x \leqslant \alpha(t) \\ f(x+\sqrt{2}t) - q(x - \sqrt{2}t), & \alpha(t) \leqslant x \leqslant \sqrt{2}t \\ f(x+\sqrt{2}t), & x \geqslant \sqrt{2}t \end{cases}$$

where $x = \alpha(t)$ is the shock front, and $2\alpha(\xi_0/2\sqrt{2}) = \xi_0$. The functions $\alpha(t)$, $q(\eta)$ $(q(0) = 0)$ are desired. We have for them from the relationships (6.2)

$$\begin{aligned} &f(\alpha(t) - \sqrt{2}t) = f(2\alpha(t)) + (\sqrt{2}t - \alpha(t))\,f'(2\alpha(t)) \\ &q(\alpha(t) - \sqrt{2}t) = (\sqrt{2}t - \alpha(t))\,f'(2\alpha(t)) \end{aligned} \tag{10.3}$$

By the theorem of implicit functions $\alpha(t)$ is found from the first equation in (10.3), and $q(\eta)$ is found from the second. The complexity of the first equation in (10.3) is that if $f''(\xi_0) = 0$ and $f'''(\xi_0) \neq 0$, this equation has two solutions. One, $\alpha(t) = \sqrt{2}t$, is not suitable. To find $\alpha'(\xi_0/2\sqrt{2})$ for the second solution, the first equation in (10.3) must be differentiated thrice and we must set $t = \xi_0/2\sqrt{2}$. We hence obtain $\alpha'(\xi_0/2\sqrt{2}) = -\sqrt{2}/5$.

Therefore, in the case under consideration, the compression wave moving to the left is transformed into a rarefaction wave of the semisignoton type on being reflected from a free edge. Then at a time $t = \xi_0/2\sqrt{2}$ a shock originates on the front of this semisignoton and moves to the left at a velocity $-\sqrt{2}/5$ at a time $t = \xi_0/2\sqrt{2}$.

Note that this velocity is independent of the kind of function f.

11. Motion of particles of an elastically granular medium in a gravity force field. 1°. *Lift of a compression wave and its reflection from a free surface.* The x-axis in this problem is assumed to be directed downward. We consider equation (0.3) for which $A > 0$ and $x \geqslant 0$ and we take its solution for $t < 0$ in the following form:

$$u(x,t) = \begin{cases} -Ax^2/4, & 0 \leqslant x \leqslant -\sqrt{2}t \\ -Ax^2/4 + f(\sqrt{2}t + x), & x \geqslant -\sqrt{2}t \end{cases}$$

where $f(0) = 0$, $f'(\xi) \leqslant 0$ for $\xi \geqslant 0$. The edge $x = 0$ is assumed to be free, i.e., $u_x(0,t) \geqslant 0$. Thus, for $t < 0$ a compression wave is propagated from the bottom upward (against the direction of the gravity force), and reaches the free surface at $t = 0$.

First, let $|f''(\xi)| \leqslant A/4$. In this case, for $t > 0$ the solution agrees with the solution of the linear problem and has the form

$$u(x,t) = \begin{cases} f(x + \sqrt{2}t) + f(\sqrt{2}t - x) - Ax^2/4, & 0 \leqslant x \leqslant \sqrt{2}t \\ f(\sqrt{2}t - x) - Ax^2/4, & x \geqslant \sqrt{2}t \end{cases}$$

Indeed, in this case we have for $0 \leqslant x \leqslant \sqrt{2}t$

$$u_x = 2x(f''(\lambda) - A/4) < 0$$

Now, let $f''(\xi)$ be a monotonically decreasing function and let there be a $x_1 > 0$ for which $f''(2x_1) = A/4$. In this case, for $0 \leqslant t \leqslant t_1 = x_1/\sqrt{2}$ the solution has the form

$$u(x,t) = \begin{cases} {}^1\!/_2 A(t^2 + x^2 - 2\sqrt{2}xt) + f(2x) + 2(\sqrt{2}t - x)f'(2x), \\ \qquad 0 \leqslant x \leqslant \sqrt{2}t \\ f(x - \sqrt{2}t) - Ax^2/4, \quad x \geqslant \sqrt{2}t \end{cases}$$

In fact, for $0 \leqslant x \leqslant \sqrt{2}t \leqslant x_1$ the inequality

$$u_x = 4(\sqrt{2}t - x)(f''(2x) - A/4) > 0$$

is satisfied.

For $t \geqslant t_1$ we shall seek the solution in the form

$$u(x,t) = \begin{cases} {}^1\!/_2 A(t^2 + x^2 - 2\sqrt{2}xt) - f(2x) + 2(\sqrt{2}t - x)f'(2x), \\ \qquad 0 \leqslant x \leqslant \alpha(t) \\ f(\sqrt{2}t + x) - r(\sqrt{2}t - x) - Ax^2/4, \quad \alpha(t) \leqslant x \leqslant \sqrt{2}t \\ f(\sqrt{2}t + x) - Ax^2/4, \quad x \geqslant \sqrt{2}t \end{cases}$$

where $x = \alpha(t)$ is the shock front.

We find equations for $\alpha(t)$ and $r(\xi)$ from the relationships (6.2)

$$\begin{aligned} f(\alpha(t) - \sqrt{2}t) &= f(2\alpha(t)) + (\sqrt{2}t - \alpha(t))f'(2\alpha(t)) + \\ &\quad {}^1\!/_8 A(\alpha(t) - \sqrt{2}t)^2 \\ r(\sqrt{2}t - \alpha(t)) &= (\sqrt{2}t - \alpha(t))(f'(2\alpha(t)) - {}^1\!/_2 A\alpha + \\ &\quad {}^1\!/_8 A(\sqrt{2}t - \alpha(t))) \end{aligned} \tag{11.1}$$

By the theorem on implicit functions, $\alpha(t)$ is found from the first equation in (11.1), and $r(\xi)$ is determined from the second equation in (11.1). Let $f'''(2x_1) \neq 0$ then by differentiating the first equation in (11.1) three times with respect to t and setting $t = t_1$ we find that $\alpha'(t_1) = -\sqrt{2}/5$.

To complete the investigation of the solution in the neighbourhood of $t = t_1$ it remains to confirm that $u_x < 0$ for $\alpha(t) \leqslant x \leqslant \sqrt{2}t$. To prove this inequality it is sufficient to show that $u_x(\sqrt{2}t - 0, t) < 0$ for $t > t_1$. We find from the second equation in (11.1) that

$$u_x(\sqrt{2}t - 0, t) = 2(\sqrt{2}t - x_1)(f''(\lambda) - A/4), \quad 2x_1 < \lambda < 2\sqrt{2}t$$

and since $f''(\lambda) - A/4 < 0$, then $u_x(\sqrt{2}t - 0, t) < 0$.

Thus, in the problem under consideration the compression wave is reflected from the free surface and is transformed into a rarefaction wave of the semisignoton type moving downward. Then, at the time $t_1 = x_1/\sqrt{2}$ a shock moving upward originates at the semisignoton front, the magnitude of the shock veloicty at the time of origination here equals $\sqrt{2}/5$, i.e., is independent of either f or A.

Further investigation of the evolution of the solution for $t > t_1$ is made difficult in that in the general case it is impossible to find the function $x = \alpha(t)$ explicitly from the first equation in (11.1). Consequently, we will examine a specific function

$$f(\xi) = \begin{cases} -\xi^3 + 3\xi^2 - 3\xi, & 0 \leqslant \xi \leqslant 1 \\ -1, & \xi \geqslant 1 \end{cases}$$

In this case the shock originates if $A < 24$. Here $x_1 = 1/2 - A/48$, $\alpha(t) = (-\sqrt{2}t + 6x_1)/5$. Furthermore, it is confirmed directly that the shock reaches the free edge and is reflected by the compression wave.

2°. *Incidence of a rarefied system on a rigid foundation in a gravity force field.* The x-axis is assumed to be directed upward. In this case $0 \leqslant x \leqslant l$, $A < 0$, $u(x, 0) = u_0(x)$, $u_t(x, 0) = v_0(x)$. The solution for $t \leqslant 0$ is taken in the form

$$u(x, t) = u_0(x) + tv_0(x) + At^2/2$$

where $u_x(x, t) \geqslant 0$, $x + u(x, t) \geqslant 0$. For $t \geqslant 0$ the solution is sought in the form

$$u(x, t) = \begin{cases} p(x + \sqrt{2}t) - p(\sqrt{2}t - x) - Ax^2/4, & 0 \leqslant x \leqslant \alpha(t) \\ u_0(x) + tv_0(x) - At^2/2, & \alpha(t) \leqslant x \leqslant l \end{cases}$$

where $x = \alpha(t)$ is the shock front. We find equations for the unknown functions $\alpha(t)$ and $p(\xi)$ from the relationships (6.2)

$$\pm 2\sqrt{2}p(\sqrt{2}t \pm \alpha) = \sqrt{2}(u_0(\alpha) + tv_0(\alpha)) \pm V_0(\alpha) + {}^1/_4 A\sqrt{2}(\alpha \pm \sqrt{2}t)^2 \qquad (11.2)$$

We consider the system (11.2) in the special case when

$$u_0(x) = kx^2, \qquad v_0(x) = -Vx$$

In this case for $t < 0$ the solution has the form

$$u(x, t) = kx^2 - Vxt + At^2/2$$

The condition of no signoton is ensured for $t > 0$ by the inequality $V < 2\sqrt{2k}$. The solution for $t > 0$ is a homogeneous function of the second degree, i.e., $p(\xi) = p\xi^2$, $\alpha(t) = \alpha t$. From (11.2) we have

$$A(\alpha^2 - 2)^2 + 4k\alpha^4 + 8(k\alpha^2 - V\alpha) = 0$$

We set $\alpha = \sqrt{2}/2$; then A, k, V are connected by the relationship $9A + 20k - 16\sqrt{2}V = 0$. We set $A = -4k/3$, $V = \sqrt{2}k/4$.

Thus, for $0 \leqslant t \leqslant \sqrt{2}l$ the solution is determined by the formulas

$$u(x, t) = \begin{cases} -\dfrac{7\sqrt{2}}{12}kxt + \dfrac{1}{3}kx^2, & 0 \leqslant x \leqslant \sqrt{2}t \\ kx^2 - \dfrac{\sqrt{2}}{4}kxt - \dfrac{2}{3}kt^2, & \sqrt{2}t \leqslant x \leqslant l \end{cases}$$

If $\sqrt{2}l \leqslant t \leqslant 3\sqrt{2}l/2$, then

$$u(x, t) = \begin{cases} -\dfrac{7\sqrt{2}}{12}kxt + \dfrac{1}{3}kx^2, & 0 \leqslant x \leqslant 3l - \sqrt{2}t \\ \dfrac{5}{8}kx^2 + \dfrac{7}{12}kt^2 + \dfrac{9}{8}kl^2 - \dfrac{5}{4}kl(x + \sqrt{2}t), & 3l - \sqrt{2}t \leqslant x \leqslant l \end{cases}$$

If $3\sqrt{2}l/2 \leqslant t \leqslant 9\sqrt{2}l/5$, then

$$u(x, t) = \begin{cases} \dfrac{7\sqrt{2}}{12}kxt - \dfrac{5}{2}klx + \dfrac{1}{3}kx^2, & 0 \leqslant x \leqslant \sqrt{2}t - 3l \\ \dfrac{5}{8}kx^2 + \dfrac{7}{12}kt^2 - \dfrac{5}{4}kl(x + \sqrt{2}t) - \dfrac{9}{8}kl^2, & \sqrt{2}t - 3l \leqslant x \leqslant l \end{cases}$$

For $9\sqrt{2}l/5 \leqslant t \leqslant 2\sqrt{2}l$ the solution is determined by the equalities

$$u(x, t) = \begin{cases} \dfrac{7\sqrt{2}}{12}kxt - \dfrac{5}{2}klx + \dfrac{1}{3}kx^2, & 0 \leqslant x \leqslant \dfrac{21}{5}l - \sqrt{2}t \\ p(x + \sqrt{2}t) + q(x - \sqrt{2}t) + \dfrac{kx^2}{3}, & \dfrac{21}{5}l - \sqrt{2}t \leqslant x \leqslant \alpha(t) \\ -\dfrac{5}{4}kx^2 - \dfrac{15}{2}kxl - \dfrac{3}{4}kl^2 - \dfrac{2}{3}kt^2 + \dfrac{5\sqrt{2}}{4}kt(2x + l), & \\ \alpha(t) \leqslant x \leqslant \sqrt{2}t - 3l & \\ \dfrac{5}{8}kx^2 + \dfrac{7}{12}kt^2 - \dfrac{5}{4}kl(x + \sqrt{2}t) - \dfrac{9}{8}kl^2, & \sqrt{2}t - 3l \leqslant x \leqslant l \end{cases}$$

where $x = \alpha(t)$ is the shock front.

We shall consider $q(-3l) = 0$, then

$$p\left(\frac{2l}{5}l\right) = -\frac{6}{25}kl^2, \quad q(\eta) = -\frac{7}{48}k\eta^2 - \frac{5}{4}kl\eta - \frac{39}{16}kl^2$$

We find the functions $p(\xi), \alpha(t)$ from the relationships (6.2). For $\alpha(t)$ we have

$$2q(\alpha - \sqrt{2}t) = -\frac{5}{2}k\alpha^2 - \frac{35}{4}kl\alpha - \frac{3}{4}kl^2 + \frac{5\sqrt{2}}{4}kt(2\alpha + l) - \frac{1}{3}k(\alpha - \sqrt{2}t)^2$$

Hence

$$\alpha = \frac{\sqrt{2}}{61}t + \frac{33}{61}l$$

For the function $p(\xi)$ we have the equation

$$2p(\alpha + \sqrt{2}t) = -\frac{25}{4}kl\alpha - \frac{3}{4}kl^2 + \frac{5\sqrt{2}}{4}k(2\alpha + l) - \frac{1}{3}k(\alpha + \sqrt{2}t)^2$$

The behaviour of the solution for large values of t can be investigated within the framework of analogous constructions. However, the calculations in this specific problem become extremely awkward.

We write the qualitative structure of the solution of the last problem for $0 \leqslant t \leqslant 2\sqrt{2}l$. For $t = 0$ a shock originates at $x = 0$ and moves upward at the velocity $\sqrt{2}/2$. For $0 \leqslant x \leqslant \sqrt{2}t/2$ the medium is compressed, $u(0, t) = 0$ and for $\sqrt{2}t/2 \leqslant x \leqslant l$ the medium is rarefied. This structure of the solution holds for $0 \leqslant x \leqslant \sqrt{2}l/2$. If $\sqrt{2}l/2 \leqslant t \leqslant 9\sqrt{2}l/5$ then the medium is in the compressed state: $u(0, t) = 0, u_x(l, t) = 0$. If $9\sqrt{2}l/5 \leqslant x \leqslant 2\sqrt{2}l$ then at the point $x = 3l/5$ a semisignoton ${}^0\gamma_-$ and a shock α, moving upward at a velocity $\sqrt{2}/61$ originate at $t = 9\sqrt{2}l/5$.

Therefore, the medium is rarefied between the shock and semisignoton fronts in the time interval under consideration, and is in the compressed state the rest of the time. Note that the velocity of the orignating shock is low.

REFERENCES

1. LOMAKIN E.V. and ROBOTNOV YU.N., Elasticity theory relationships for an isotropic body of different moduli, Izv. Akad. Nauk SSSR, Mekhan. Tverd. Tela, No.6, 1978.
2. AMBARTSUMYAN S.A., Equations of the plane different-resistivity or different-moduli theory of elasticity, Izv. AKad. Nauk ArmSSR, Mekhanika, Vol.19, No.2, 1966.
3. ROZHDESTVENSKII B.L. and YANENKO N.N., Systems of Quasilinear Equations and Their Application to Gas Dynamics. Nauka, Moscow, 1978.
4. JOHN F., Formation of singularities in one-dimensional non-linear wave propagation. Communs. Pure Appl. Math., Vol.27, No.3, 1974.
5. GLIMM J., Solution in the large for non-linear hyperbolic systems of equations. Communs. Pure Appl. Math., Vol.18, No.4, 1965.
6. SEDOV L.I., Mechanics of a Continuous Medium, Vol.2, Nauka, Moscow, 1970.
7. ZEL'DOVICH YA.B. and RAIZER YU.P., Physics of Shocks and High-Temperature Hydrodynamic Phenomena. Fizmatgiz, Moscow, 1963.
8. KNOPOFF L., Elastic wave damping in the earth. Physical Acoustics, Vol.3, Pt.B. Dynamics of arrays. Mir, Moscow, 1968.
9. GOL'DENBLAT I.I., Non-linear Problems of Elasticity Theory. Nauka, Moscow, 1969.
10. LAX P.D., Hyperbolic systems of conservation laws. II. Communs. Pure Appl. Math. Vol.10, No.4, 1957.
11. MASLOV V.P. and MOSOLOV P.P., Vibrations of rods of different moduli, Uspekhi Matem. Nauk, Vol.36, No.3, 1981.
12. ARNOL'D V.I., VARCHENKO A.N. and GUSEIN-ZADE S.M., Singularities of Differentiable Mappings, Vol.1, Nauka, Moscow, 1982.

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